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TWO DIMENSIONAL UNITAL RIESZ ALGEBRAS, THEIR REPRESENTATIONS AND NORMS

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Abstract. We describe all two dimensional unital Riesz algebras and study representations of them in Riesz algebras of regular operators. Although our results are not complete, we do demonstrate that very varied behaviour can occur even though all these algebras can be given a Banach lattice algebra norm.

1. Introduction.

A *Riesz algebra* is an associative algebra over the reals that is simultaneously a Riesz space with the two structures connected by the implication $x, y \geq 0 \implies xy \geq 0$. We will be concerned only with Archimedean Riesz algebras and almost all of the time we assume the existence of a multiplicative identity. Without some rather special assumptions, little is known about these objects. In this work we study what is probably the simplest non-trivial class of such algebras, namely the two-dimensional examples. We will see that even here there are interesting aspects to their study and unsolved problems. Given that there is, up to an order isomorphism, only one two-dimensional Archimedean Riesz space and, up to an algebra isomorphism, only three two-dimensional real associative algebras, the reader would be forgiven for thinking that there are not many algebras for us to study. In fact, the different possible relationships between the two structures give us plenty of examples to consider.

Up to an algebra isomorphism the three associative algebra structures on $\mathbb{R}^2$ are the *complex numbers*, where $(x, y)(x', y') = (xx' - y'y, xy' + x'y)$, the *split complex numbers*, where $(x, y)(x', y) = (xx' + y'y, xy' + x'y)$ and the *dual numbers*, where $(x, y)(x', y) = (xx', xy' + x'y)$. The earliest explicit reference that we know of for this fact is in [8], although this was surely known long before Yaglov's work. Of course, using different bases for $\mathbb{R}^2$ leads to different looking descriptions of these algebras. For example using the basis $\{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})\}$ the

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split complex number system simply corresponds to coordinate-wise multiplication. At various times we will find it convenient to switch between different descriptions of our algebras depending on whether we want the multiplicative or the order structure to take on a simple form. It will be rare that both are possible at the same time.

An Archimedean order on $\mathbb{R}^2$ is determined by its positive cone and that order is automatically a lattice ordering. We will be able to describe all the algebras of interest to us using cones of the form

$$P_\alpha^\beta = \{(x, y) : x \geq 0, \alpha x \leq y \leq \beta x\}$$

for $\alpha, \beta \in \mathbb{R} \cup \{-\infty, \infty\}$, $\alpha < \beta$, where we interpret, for example, $y \leq \infty x$ as being always true. Thus the standard lattice order on $\mathbb{R}^2$ is that induced by the cone P_0^∞. We will denote the multiplication on $\mathbb{R}^2$ that makes it isomorphic to the complex numbers by $\times_{\mathbb{C}}$; that making it isomorphic to the the split complex numbers by $\times_{\mathbb{S}}$ and that for the dual numbers by $\times_{\mathbb{D}}$. Later we will introduce other notations to describe the same multiplications relative to different bases of $\mathbb{R}^2$. In view of the risk of confusion caused by our multiple representations, we will always spell out in full the algebras that we are considering, for example the complex numbers under the coordinate-wise order (which is *not* a Riesz algebra) is $(\mathbb{R}^2, \times_{\mathbb{C}}, P_0^\infty)$.

We emphasise that the positive cone P in a Riesz space X is always *proper* or *pointed* in that $P \cap (-P) = \{0\}$ and that a cone in $\mathbb{R}^2$ (indeed in any finite dimensional vector space) induces an Archimedean order if and only if it is closed for the usual Euclidean topology.

We will look at the three multiplicative structures in turn and characterize the possible Archimedean Riesz space orders for which we obtain a Riesz algebra. I.e. those positive cones which are closed under multiplication. Our results to this point were certainly known to Birkhoff and Pierce, [4] Examples 9e and 9f, although they give no proofs.

The bulk of our work consists of our investigations into when our examples have representations, in various senses, in an algebra of regular operators on a Riesz space. Although not complete, our results both show a surprising variation in possible behaviours and suggest some open problems in the study of regular operators.

We conclude with a brief section showing that all our examples of two dimensional unital Riesz algebras my be given norms which are both lattice norms and which are sub-multiplicative. However, some of our examples have a multiplicative identity which is not positive so by Theorem 2 of [3] we cannot also require that this identity have norm one.

We assume that the reader is familiar with the basics of Riesz space theory, but suggest [1] as a good source for any terms that the reader may be unfamiliar with. As we use them several times in this paper, it might be worth reminding readers of the *Riesz-Kantorovich* formulae. Namely, that if X is an ordered vector space with the Riesz decomposition property and Y a Dedekind complete Riesz space then the order bounded operators from X into Y form a Dedekind complete Riesz space under the usual operator order and furthermore if $T : X \to Y$ is order bounded and $x \in X_+$ then $T^+(x) = \sup\{Ty : 0 \leq y \leq x\}$. Several other similar formulae hold, for which we refer the reader to Theorem 1.59 of [1]. In this context it might be worth pointing out that $\mathbb{R}^2$, under any generating Archimedean order is necessarily a Dedekind complete Riesz space and therefore has the Riesz decomposition property.

2. Riesz algebra orderings on the dual numbers

We start by determining the closed cones under which the dual numbers are a Riesz algebra.

Proposition 2.1. *The closed cones $P \subset \mathbb{R}^2$ which make $(\mathbb{R}^2, \times_{\mathbb{D}}, P)$ into a Riesz algebra are the cones $P = P_\alpha^\infty$ for $\alpha \in [0,\infty)$ and $P = P_{-\infty}^\beta$ for $\beta \in (-\infty, 0]$.*

Proof. We show first that each cone P_α^∞, for $\alpha \in [0,\infty)$ is closed under multiplication and therefore does give us a Riesz algebra. If $(x,y),(x',y') \in P_\alpha^\infty$ then we have $y \geq \alpha x \geq 0$ and $y' \geq \alpha x' \geq 0$. It follows that $(x,y) \times_{\mathbb{D}} (x',y') = (xx', xy' + x'y) \in P_\alpha^\infty$ as $xy' + x'y \geq x(\alpha x') + x'(\alpha x) = 2\alpha xx' \geq \alpha xx' \geq 0$, given that $\alpha \geq 0$. As similar argument shows that each $P_{-\infty}^\beta$ is closed under multiplication.

Now suppose that $(\mathbb{R}^2, \times_{\mathbb{D}}, P)$ is a Riesz algebra. First note that $(-1,0) \notin P$ else $(-1,0) \times_{\mathbb{D}} (-1,0) = (1,0) \in P$ and P would not be a proper cone. Therefore P cannot contain any element of the form $(x,0)$ for $x < 0$. Now suppose that $(x,y) \in P$ where $x < 0$ and $y < 0$ and therefore $(x,y) \times_{\mathbb{D}} (x,y) = (x^2, 2xy) \in P$. With the notation of Figure 1, $\theta = \tan^{-1}(2y/x) > \phi = \tan^{-1}(y/x)$ so the cone must contain the region to the left and above the two lines as otherwise, by convexity, it would be the whole of $\mathbb{R}^2$. It follows that $(x,0) \in P$ which we have already seen to be impossible. A similar argument shows that it is impossible for $(x,y) \in P$ with $x < 0$ and $y > 0$.

Thus $P \subseteq \{(x,y) : x \geq 0\}$ and therefore must be one of the cones P_α^β for $\alpha, \beta \in \mathbb{R} \cup \{-\infty, \infty\}, \alpha < \beta$ and either $\alpha \neq -\infty$ or $\beta \neq \infty$. If $\beta < \infty$ then $(1,\beta) \in P$ and hence its n'th power under $\times_{\mathbb{D}}$, $(1, n\beta)$,

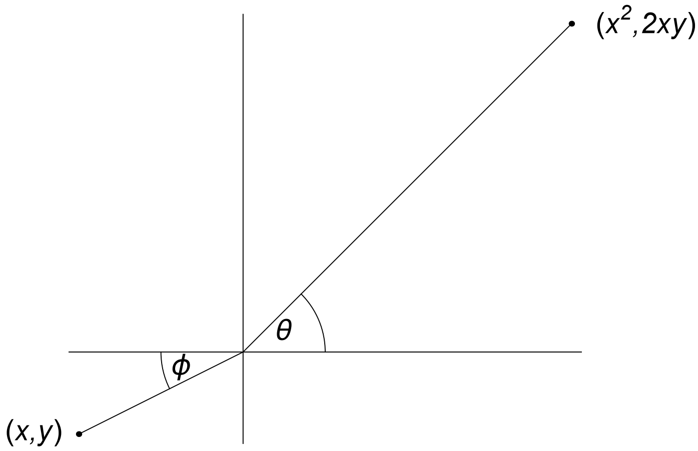

FIGURE 1

lies in P. We thus have $n\beta \leq \beta$ for all $n \in \mathbb{N}$ showing that $\beta \leq 0$. Similarly, if $\alpha > -\infty$ then $(1, \alpha) \in P$ and hence $(1, n\alpha) \in P$ for all $n \in \mathbb{N}$ and $\alpha \geq 0$. As $\alpha < \beta$ this leaves only the possibilities that $\alpha = -\infty$ and $\beta \leq 0$ or else $\alpha \geq 0$ and $\beta = \infty$.

□

Not all of these cones give rise to non-isomorphic Riesz algebra structures. We will say that two Riesz algebras A and B are *isomorphic* if there is a linear bijection between them that is simultaneously a unital algebra isomorphism and an order isomorphism.

Proposition 2.2. *Up to isomorphism there are two Riesz algebra orders on* $(\mathbb{R}^2, \times_{\mathbb{D}})$.

Proof. If $\alpha, \alpha' > 0$ then $A_\alpha = (\mathbb{R}^2, \times_{\mathbb{D}}, P_\alpha^\infty)$ and $A_{\alpha'} = (\mathbb{R}^2, \times_{\mathbb{D}}, P_{\alpha'}^\infty)$ are isomorphic, as the map $\pi : (x, y) \mapsto (x, \alpha' y/\alpha)$ is clearly an order isomorphism and is a unital algebra isomorphism as

$$\begin{aligned}
\pi\big((x,y)\big) \times_{\mathbb{D}} \pi\big((x',y')\big) &= (x, \alpha' y/\alpha) \times_{\mathbb{D}} (x', \alpha' y'/\alpha) \\
&= \big(xx', (\alpha' y/\alpha)x' + x(\alpha' y'/\alpha)\big) \\
&= \big(xx', \alpha'(xy' + x'y)/\alpha\big) \\
&= \pi\big((xx', xy' + x'y)\big) \\
&= \pi\big((x,y) \times_{\mathbb{D}} (x',y')\big)
\end{aligned}$$

and the identity $(1, 0)$ is clearly fixed under π. Similarly all the Riesz algebras $B_\beta = (\mathbb{R}^2, \times_{\mathbb{D}}, P_{-\infty}^\beta)$, for $\beta < 0$ are isomorphic. The map $\sigma : (x, y) \to (x, -y)$ is also easily checked to be an isomorphism of A_α

onto $B_{-\alpha}$ for $\alpha > 0$. The map σ is also an isomorphism of A_0 onto B_0, so that we have at most two isomorphism classes. But in A_0 the multiplicative identity is positive, whilst in A_α for $\alpha > 0$ it is not, so these are not isomorphic and we do have two non-isomorphic Riesz algebra structures. □

3. Riesz algebra orderings on the split complex numbers

As we noted above, by a change of basis we may identify the split complex numbers with $\mathbb{R}^2$ under pointwise multiplication, which we will denote by an unadorned multiplication sign, so that $(x,y) \times (x',y') = (xx',yy')$. The simplification that this gives to discussions of the multiplication is enough for us to look at this description of the split complex numbers throughout this section.

Theorem 3.1. *The closed cones $P \subset \mathbb{R}^2$ which make $(\mathbb{R}^2, \times, P)$ into a Riesz algebra are:*

(1) P_0^∞;
(2) P_α^β *for* $0 < \beta \le 1$ *and* $-\sqrt{\beta} \le \alpha \le 0$;

and their images under the map $(x,y) \mapsto (y,x)$.

Proof. Note that the sets claimed as the possible positive cones are indeed proper cones. We start by showing that these cones are closed under multiplication. P_0^∞ is just the standard cone which is certainly closed under multiplication. If $0 < \beta \le 1$, $-\sqrt{\beta} \le \alpha \le 0$ and $(x,y),(x',y') \in P_\alpha^\beta$ then we have $x,x' \ge 0$, $\alpha x \le y \le \beta x$ and $\alpha x \le y' \le \beta x'$. Certainly $xx' \ge 0$. The remainder of this proof we split into three cases:

(i) If $y,y' \ge 0$ then
$$\alpha xx' \le 0 \le yy' \le \beta^2 xx' \le \beta xx',$$
as $0 < \beta \le 1$, so that $(x,y) \times (x',y') = (xx',yy') \in P_\alpha^\beta$.

(ii) If $y,y' \le 0$ then
$$\alpha xx' \le 0 \le yy' \le \alpha^2 xx' \le \beta xx',$$
as $-\sqrt{\beta} \le \alpha \le 0$, so that $(x,y) \times (x',y') = (xx',yy') \in P_\alpha^\beta$.

(iii) If $y > 0$ and $y' < 0$ (say) then
$$yy' - (\beta x)(\alpha x') = y(y' - \alpha x') + (y - \beta x)(\alpha x') \ge 0,$$
as $y, y' - \alpha x' \ge 0$ and $y - \beta x, \alpha x' \le 0$. Thus
$$\beta xx' \ge 0 > yy' \ge (\beta x)(\alpha x') = (\alpha\beta)(xx') \ge \alpha xx',$$

as $\beta \le 1$ and $\alpha \le 0$, so that again $(x,y)\times(x',y') = (xx',yy') \in P_\alpha^\beta$. Clearly interchanging x and y will also produce Riesz algebra cones.

Now suppose that P is a Riesz algebra cone for $(\mathbb{R}^2,\times)$. If $x<0$ and $(x,0)\in P$ then $(x,0)\times(x,0) = (x^2,0)\in P$ and P would not be a proper cone. Similarly, if $y<0$ then $(0,y)\notin P$. It follows, by convexity, that if $(x,y)\in P$ with $x,y<0$ then P must be contained in the set $\{(x,y) : x<0, y<0\}$. But this is contradicted by the requirement that $(x^2,y^2)\in P$. Thus P cannot contain any point (x,y) with both x and y negative. It follows from this that P cannot contain points (x,y) and (x',y') with $x<0, y>0, x'>0$ and $y'<0$ as then $(xx',yy')\in P$ and both xx' and yy' are negative. Thus we either have $P\subset\{(x,y) : x\ge 0\}$ or $P\subset\{(x,y) : y\ge 0\}$. We will deal with the first possibility and the second follows from an interchange of coordinates.

Thus we must determine which of the cones P_α^β are closed under $\times$. The first thing to note is that we cannot have $\alpha=-\infty$. Indeed if $P_{-\infty}^\beta$ were an algebra cone then chose $y < -(|\beta| \vee \sqrt{|\beta|})$ then $y<\beta$ so that $(1,y)\in P_{-\infty}^\beta$ but $y^2>\beta$ so that $(1,y^2) = (1,y)\times(1,y)\notin P_{-\infty}^\beta$.

If P_α^∞ is an algebra cone then we must have either $\alpha=0$ or $\alpha\ge 1$. As $(1,\alpha)\in P_\alpha^\infty$ and $(1,n)\in P_\alpha^\infty$ for all sufficiently large integers n, $(1,\alpha)\times(1,n) = (1,n\alpha)\in P_\alpha^\infty$ and therefore $n\alpha\ge\alpha$ for large n, which implies that $\alpha\ge 0$. On the other hand, $(1,\alpha)\times(1,\alpha) = (1,\alpha^2)\in P_\alpha^\infty$ so that $\alpha^2\ge\alpha$. It follows that either $\alpha=0$ or $\alpha\ge 1$. When $\alpha=0$ we have case (1), whilst if $\alpha\ge 1$ the resulting algebra is isomorphic, *via* the mapping $(a,b)\mapsto(b,a)$, to the algebra obtained from case (2) with the cone $P_0^{\alpha^{-1}}$.

Finally, suppose that $\alpha,\beta\in\mathbb{R}$ and that P_α^β is an algebra cone. As $(1,\beta)\in P_\alpha^\beta$, we must have $(1,\beta)\times(1,\beta) = (1,\beta^2)\in P_\alpha^\beta$ and therefore $\beta^2\le\beta$, from which it follows that $0\le\beta\le 1$. As $\alpha<\beta\le 1$ we can find $y\in(\alpha,\beta)$ with $0\le y<1$ and $(1,y)\in P_\alpha^\beta$. Therefore $(1,y^n)\in P_\alpha^\beta$ for all $n\in\mathbb{N}$ and (as $|y|<1$ and P_α^β is closed) $(1,0)\in P_\alpha^\beta$, showing that $\alpha\le 0$. As $(1,\alpha)\in P_\alpha^\beta$ we also have $(1,\alpha)\times(1,\alpha) = (1,\alpha^2)\in P_\alpha^\beta$, showing that $\alpha^2\le\beta$. This is all that we need to establish to show that we are in case (3), except for the observation that if $\beta=0$ then also $\alpha=0$ and P_α^β is not a generating cone.

□

As the only unital algebra isomorphisms of $(\mathbb{R}^2,\times)$ are the identity and the map $(x,y)\mapsto(y,x)$, all the Riesz algebras defined by the cones explicitly listed in Theorem 3.1 are not isomorphic, for differing choices of α and β.

4. Riesz algebra orderings on the complex numbers

This subsection will be very short as there is no appropriate order on $(\mathbb{R}^2, \times_{\mathbb{C}})$ under which it is a Riesz algebra.

Proposition 4.1. *There is no Archimedean order on* $(\mathbb{R}^2, \times_{\mathbb{C}})$ *for which the positive cone is closed under multiplication.*

Proof. Suppose that P were such a cone. Consider its intersection with the unit circle, $T = \{\big(\cos(\theta), \sin(\theta)\big) : \theta \in [0, 2\pi)\}$. As P is generating, $P \cap T$ has non-empty interior in T, so there is an irrational real λ with $\big(\cos(\lambda\pi), \sin(\lambda\pi)\big) \in P \cap T$. The positive integer powers of this element of $(\mathbb{R}^2, \times_{\mathbb{C}})$ are dense in T. If all these powers lay in P then $T \subset P$, as P is closed, so that P is the whole of $\mathbb{R}^2$ and is therefore not a proper cone. $\square$

5. Representations

One of the most important examples of a Riesz algebra is the algebra of all regular operators, $\mathcal{L}^r(E)$, on a Dedekind complete Riesz space E. These are the linear operators in the (real) linear span of the positive operators, where positive means that $x \geq 0 \Rightarrow Tx \geq 0$. If E is $\mathbb{R}^n$ with the coordinate-wise order then all linear operators on E are regular and an operator is positive if and only if its matrix with respect to the standard basis has all its entries non-negative. All the naturally occurring examples of Riesz spaces, and this is especially true when a normed space structure is added in, may be identified with subsets of some $\mathcal{L}^r(E)$ that are both subalgebras and sublattices. A little more generality may be expected by allowing E not to be Dedekind complete, when $\mathcal{L}^r(E)$ will not usually be a Riesz space, provided we now talk about subsets H that are in themselves Riesz spaces and where any pair $S, T \in H$ which has a supremum in $\mathcal{L}^r(E)$ has the same supremum in H. This level of generality will not actually be needed in this work so we will not pursue this further at present.

Our expectation from the theory of unital algebras would be that the left regular representation will provide us with the representation that we want, at least in the simple case that the Riesz algebra that we start with is Dedekind complete. For example Schep, in Proposition 1 of [7], shows that if we start with a Dedekind complete Banach lattice E then the left regular representation of $\mathcal{L}^r(E)$ does give a subspace of $\mathcal{L}^r\big(\mathcal{L}^r(E)\big)$ which is isometrically algebra and lattice isomorphic to $\mathcal{L}^r(E)$ itself. This turns out not usually to be the case, even in our current very simple setting.

One problem that we will certainly encounter arises because not all of our examples of two-dimensional unital Riesz algebras have a positive multiplicative identity. If a representation in some $\mathcal{L}^r(E)$ were unital then the identity would map to the identity operator on E, which is positive, so we immediately lose even an order isomorphism, let alone being able to expect a lattice isomorphism. Consequently, we need to be rather careful with our definitions.

Definition 5.1. If $(A, \bullet, P)$ is a Riesz algebra and E a Dedekind complete Riesz space, then, using $\circ$ to denote composition in $\mathcal{L}^r(E)$,

(1) The linear map $\pi : A \to \mathcal{L}^r(E)$ is an *order representation* if $\pi(a \bullet b) = \pi(a) \circ \pi(b)$ for all $a, b \in A$ and $a \in P \Leftrightarrow \pi(a) \geq 0$. If A has a multiplicative identity E and $\pi(e)$ is the identity operator in $\mathcal{L}^r(E)$ then we refer to a *unital order representation.*
(2) The linear map $\pi : A \to \mathcal{L}^r(E)$ is an *lattice representation* if $\pi(a \bullet b) = \pi(a) \circ \pi(b)$ and $\pi(a \vee b) = \pi(a) \vee \pi(b)$ for all $a, b \in A$. If A has a multiplicative identity E and $\pi(e)$ is the identity operator in $\mathcal{L}^r(E)$ then we refer to a *unital lattice representation.*

Note that, as all the cones involved are generating, an order representation is automatically injective as $a = 0 \Leftrightarrow -a, a \in P \Leftrightarrow -\pi(a), \pi(a) \geq 0 \Leftrightarrow \pi(a) = 0$. Observe also that a lattice representation need not be an order isomorphism, but that faithful lattice representations (i.e. injective lattice representations) are.

If $(A, \bullet, P)$ is a Dedekind complete Riesz algebra then the *left regular representation* of A is the map $\pi : A \to \mathcal{L}^r(A)$ defined by $\pi(a)(b) = a \bullet b$. It is elementary to verify that π is a linear map taking values in $\mathcal{L}^r(A)$ and that if $(A, \bullet, P)$ has a multiplicative identity then π is injective.

Proposition 5.2. *If $(A, \bullet, P)$ is a Dedekind complete Riesz algebra with a positive identity e then the left regular representation of A is a unital order representation of A in $\mathcal{L}^r(A)$.*

Proof. The only new feature here is that $\pi(a) \geq 0$ if and only if $a \in P$. Saying that $\pi(a) \geq 0$ asserts precisely that $\pi(a)(x) = a \bullet x \in P$ whenever $x \in P$ and this is an immediate consequence of A being a Riesz algebra if $a \in P$. Conversely, if $\pi(a) \geq 0$ and $e \in P$ then $\pi(a)(e) = a \bullet e = a \in P$. □

If there is a multiplicative identity which is not positive then the left regular representation certainly will not be an order representation.

If $(A, \bullet, P)$ is not unital, then the usual manner of adding an identity may be used and the order extends naturally to this. To be precise, we set $A_e = \mathbb{R} \times A$, define the multiplication by $(\lambda, a) \star (\mu, b) = (\lambda\mu, \lambda b +$

$\mu a + a \bullet b)$ and let $P_e = \mathbb{R}_+ \times P$, then $(A_e, \star, P_e)$ is a Riesz algebra with a positive multiplicative identity, namely $(1,0)$. This new Riesz algebra will be Dedekind complete if the original one was. Consequently, the left regular representation of $(A_e, \star, P_e)$ in $\mathcal{L}^r(A_e)$ will be a unital order representation. The restriction of this representation to $(A, \bullet, P)$ will also be an order representation. This process of adding a multiplicative identity works even if A has a multiplicative identity e already, although $(0, e)$ will no longer be a multiplicative identity in $(A_e, \star, P_e)$. In this case the order representation of $(A, \bullet, P)$ in $\mathcal{L}^r(A_e)$ will no longer be unital, but this does give a way to produce an order representation when there is a non-positive multiplicative identity. However, as we will see shortly, the left regular representation will rarely be a lattice representation.

6. Representations of the dual numbers

We have, up to an isomorphism, two cases to deal with.

Proposition 6.1. *The left regular representation of $A = (\mathbb{R}^2, \times_{\mathbb{D}}, P_0^\infty)$ is a faithful unital lattice representation.*

Proof. If $(a, b) \in \mathbb{R}^2$ and $\pi : A \to \mathcal{L}^r(A)$ is the left regular representation then for $(x, y) \in A$ we have

$$\pi(a, b)(x, y) = (ax, ay + bx)$$

The order on A is just the standard pointwise order so that $(a, b)^+ = (a^+, b^+)$. If $(x, y) \in P_0^\infty$ then

$$\begin{aligned}
\pi(a, b)^+(x, y) &= \sup\{\pi(a, b)(x', y') : 0 \le x' \le x, 0 \le y' \le y\} \\
&= \sup\{(ax', ay' + bx') : 0 \le x' \le x, 0 \le y' \le y\} \\
&= (a^+x, a^+y + b^+x) = \pi(a^+, b^+)(x, y),
\end{aligned}$$

so that $\pi(a, b)^+ = \pi\big((a, b)^+\big)$. □

To be specific, for the other equivalence class of Riesz algebras on the dual numbers we will take the positive cone to be $P_1^\infty = \{(x, y) : 0 \le x \le y\}$. In order to make the order structure more transparent, we make a change of basis using two extremal elements of the positive cone as basis. This converts the positive cone into the standard cone P_0^∞.

Proposition 6.2. *The Riesz algebras* $A = (\mathbb{R}^2, \times_{\mathbb{D}}, P_1^\infty)$ *and* $B = (\mathbb{R}^2, \odot, P_0^\infty)$[1] *are isomorphic, where*

$$(a,b) \odot (a',b') = (aa', ab' + a'b + aa').$$

Proof. Let $u = (1,1)$ and $v = (0,1)$ in A and define $\pi : B \to A$ by $\pi(a,b) = au + bv$. Certainly p is a linear bijection and it maps two disjoint non-zero extremal elements of the positive cone in B to disjoint non-zero extremal elements of the positive cone in A, so is an order isomorphism. We now need only verify that π preserves the multiplication structure:

$$\begin{aligned}
\pi(a,b) \times_{\mathbb{D}} \pi(a',b') &= \big(a(1,1) + b(0,1)\big) \times_{\mathbb{D}} \big(a'(1,1) + b'(0,1)\big) \\
&= (a, a+b) \times_{\mathbb{D}} (a', a'+b') \\
&= \big(aa', a(a'+b') + a'(a+b)\big) \\
&= (aa', a'b + ab' + 2aa') \\
&= aa'(1,1) + (ab' + a'b + aa')(0,1) \\
&= \pi(aa', ab' + a'b + aa') = \pi\big((a,b) \odot (a',b')\big).
\end{aligned}$$

It is easily checked directly that the multiplicative identity in A is $(1,0)$ whilst in B it is $(1,-1)$ and $\pi(1,-1) = (1,1) - (0,1) = (1,0)$ so that π is unital as well as preserving products. ☐

We know from general considerations that the left regular representation of A_e, when restricted to A, will be an order representation of A although not a unital one. It is not difficult to show that the left regular representation of A in $\mathcal{L}^r(A_e)$ is not a lattice representation. This will also follow from Corollary 7.5 below.

Nevertheless, there is a finite dimensional faithful lattice representation of A and it can be taken to act on $\mathbb{R}^3$.

Proposition 6.3. *The map* $\pi : B \to \mathcal{L}^r(\mathbb{R}^3)$ *defined by*

$$\pi(a,b) = \begin{pmatrix} a & a & b \\ 0 & 0 & a \\ 0 & 0 & a \end{pmatrix}$$

is a faithful lattice representation of B *and there is therefore a faithful lattice representation of* A.

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[1]Note that B is precisely Example 6 of [3] with $p = r = 1$. Unlike [3] where the case $0 < r < p \leq 1$ is used and the $\|\cdot\|_1$ norm is an algebra norm, the $\|\cdot\|_1$ norm is not an algebra norm here, however $\|(a,b)\| = 2|a| + |b|$ is both a lattice and algebra norm.

Proof. As π maps the disjoint atoms $(0,1)$ and $(1,0)$ in B to disjoint matrices, it is easily seen that π is a lattice homomorphism. Clearly π is injective, so it remains only to verify that π respects the multiplication. This is easily checked. We denote matrix multiplication by juxtaposition.

$$\begin{aligned}
\pi(a,b)\pi(a',b') &= \begin{pmatrix} a & a & b \\ 0 & 0 & a \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} a' & a' & b' \\ 0 & 0 & a' \\ 0 & 0 & a' \end{pmatrix} \\
&= \begin{pmatrix} aa' & aa' & ab' + a'b + aa' \\ 0 & 0 & aa' \\ 0 & 0 & aa' \end{pmatrix} \\
&= \pi(aa', ab' + a'b + aa') = \pi\big((a,b) \odot (a',b')\big).
\end{aligned}$$

$\square$

7. Representations of the split complex numbers

We saw in section 2.2 that, with one exception, the closed cones which make the split complex numbers into Riesz algebras lie in a naturally indexed family. The exceptional case simply reduces to $\mathbb{R}^2$ with the pointwise lattice and algebra operations. The left regular representation of this is clearly a faithful unital lattice representation and there is nothing more that we can add. Of the remaining cases, we look first at those for which the multiplicative identity is positive as there the left regular representation of the algebra on itself is at least a unital order representation. This is when the algebra is $(\mathbb{R}^2, \times, P^1_\alpha)$ for $-1 \leq \alpha \leq 0$.

Theorem 7.1. *If $-1 \leq \alpha \leq 0$ then there is a faithful unital lattice representation of $(\mathbb{R}^2, \times, P^1_\alpha)$ on some finite dimensional Riesz space if and only if $\alpha = -\frac{1}{n}$ for $n \in \mathbb{N}$. For such $\alpha = -\frac{1}{n}$ there is a faithful unital lattice representation of $(\mathbb{R}^2, \times, P^1_\alpha)$ on $\mathbb{R}^{n+1}$ and there is no faithful unital lattice representation on a space of smaller dimension.*

Proof. Suppose first that $\alpha > 0$. Let us write $u = (1,1), v = (1,0)$ and $w = (1,\alpha)$ so that u, v and w are positive, u and v are idempotents with $uv = vu = v$ and $u \perp w$. Also we have:

$$w = \alpha u + (1-\alpha)v.$$

If π is a faithful unital lattice representation of $(\mathbb{R}^2, \times, P^1_\alpha)$ in $\mathcal{L}^r(E)$ then $\pi u = I$, πv and πw are positive and non-zero, πv is an idempotent, $I \perp \pi w$ and $\pi w = \alpha I + (1-\alpha)\pi v$. Conversely if we can find values for

$\pi v = V$ and $\pi w = W$ related in this way then π extends to a faithful lattice representation of $(\mathbb{R}^2, \times, P^1_\alpha)$ in $\mathcal{L}^r(E)$.

Suppose now that $\alpha = -\frac{1}{n}$ with $n \in \mathbb{N}$. Let V be the $(n+1)\times(n+1)$ matrix with all entries equal to $\frac{1}{n+1}$, so that V is a positive non-zero idempotent. As $\alpha = -\frac{1}{n}$, $1-\alpha = \frac{n+1}{n}$ and we that the diagonal of $W = \alpha I + (1-\alpha)V$ is zero whilst all the off-diagonal elements are equal to $\frac{1}{n}$, so that $0 \le W \perp I$. W will be non-zero and we may produce our representation of $(\mathbb{R}^2, \times, P^1_\alpha)$ in $\mathcal{L}^r(\mathbb{R}^{n+1})$.

Suppose, on the other hand, that there is a faithful unital lattice representation of $(\mathbb{R}^2, \times, P^1_\alpha)$ on some finite dimensional space $\mathbb{R}^n$. $\pi v = V$ is an idempotent positive matrix. By Theorem 3.1 in Chapter 3 of [2], after a permutation we can write

$$V = \begin{pmatrix} J & JA & 0 & 0 \\ 0 & 0 & 0 & 0 \\ BJ & BJA & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (*)$$

where $A, B \ge 0$ and

$$J = \begin{pmatrix} J_1 & 0 & 0 & \dots & \\ 0 & J_2 & 0 & \dots & \\ & & & \ddots & \\ 0 & \dots & & & J_k \end{pmatrix}$$

where the J_i are idempotents of rank 1. In order that $\alpha I + (1-\alpha)V$ be disjoint from I, it must have diagonal zero. In particular that means that the diagonal of V must be constant so that actually $V = J$. As the rank of an idempotent matrix is equal to its trace, [5], 9.8 (2), the trace of each J_i is equal to one. For the diagonal to be constant all the J_i must have diagonal equal to $\frac{1}{n}$, for some $n \in \mathbb{N}$,. This means that $\alpha I + (1-\alpha)V$ has diagonal constantly $\alpha + (1-\alpha)\frac{1}{n}$ so that $\alpha = -\frac{1}{(n-1)}$. This argument actually shows that any faithful unital lattice representation when $\alpha = -\frac{1}{(n-1)}$ must be on a space of dimension an positive integer multiple of n.

We turn now to the case $\alpha = 0$, so that the positive cone is P^1_0. In $(\mathbb{R}^2, \times, P^1_0)$, $(1,1)$ is the positive multiplicative identity and $(1,0)$ a positive idempotent. They are disjoint elements of P^1_0. If π is a lattice representation of $(\mathbb{R}^2, \times, P^1_0)$ in $\mathcal{L}^r(\mathbb{R}^n)$ then $V = \pi(1,1)$ and $U = \pi(1,0)$ must be non-zero positive idempotent matrices with $U \circ V = V \circ U = U$ and $U \perp V$. Note first that it follows from the representation (*) above that a non-zero positive idempotent V must have a strictly positive diagonal element, for if $J = 0$ then $V = 0$, and the (non-negative) diagonal elements of each J_i sum to 1. By Proposition III.11.5 of [6], $E = V(\mathbb{R}^n)$ is a Riesz space, though not necessarily a sublattice, for the standard order inherited from $\mathbb{R}^n$. Restricting U and V to E does not destroy any of the properties that we are assuming. $V_{|E}$ is the identity and the matrix representation of $U_{|E}$ will have a strictly positive diagonal entry so that $U_{|E}$ and $V_{|E}$ are not disjoint. If $0 \leq P \neq 0$ and $P \leq U_{|E}, V_{|E}$ then $P \circ V \neq 0$ and $0 \leq P \circ V \leq U_{|E} \circ V = U, V_{|E} \circ V = V$ so that U and V are not disjoint after all. $\square$

We have had to leave questions unanswered in the last result. If $\alpha \neq -\frac{1}{n}$ for $n \in \mathbb{N}$ can there be a faithful non-unital lattice representation of $(\mathbb{R}^2, \times, P^1_\alpha)$ on a finite dimensional space? What about lattice representations, either unital or not, on infinite dimensional Riesz spaces? We suspect that there can be none, except in the one special case that we contribute here.

Proposition 7.2. *If $f \in L^1([0,1])$ define $Uf(x) = \int_0^1 f(t)\, dt$ for all $x \in [0,1]$. Then U is a non-zero positive projection on $L^1([0,1])$ that is disjoint from the identity operator and there is therefore a faithful unital lattice representation of $(\mathbb{R}^2, \times, P^1_0)$ on $L^\infty([0,1])$.*

Proof. The only statement that possibly needs proof is the assertion that U is disjoint from the identity operator I. Once that is established it is routine to check that the linear map π with $\pi(1,1) = I$ and $\pi(1,0) = U$ gives the representation that we require. Fix $n \in \mathbb{N}$ and let $h_i = \chi_{[\frac{i-1}{n}, \frac{i}{n}]}$ $(1 \leq i \leq n)$. Then, writing $\mathbf{1}$ for the constantly one function, for each i we have

$$\begin{aligned}(U \wedge I)(\mathbf{1}) &= \inf\{U(h) + I(\mathbf{1} - h) : 0 \leq h \leq \mathbf{1}\}\\ &\leq U(h_i) + I(\mathbf{1} - h_i)\\ &= \frac{1}{n}\mathbf{1} + (\mathbf{1} - h_i)\end{aligned}$$

so that $0 \leq (U \wedge I)(\mathbf{1})_{[\frac{i-1}{n}, \frac{i}{n}]} \leq 1/n$. This holds for $1 \leq i \leq n$ so that $0 \leq (U \wedge I)(\mathbf{1}) \leq 1/n$ on $[0,1]$. This holds for all $n \in \mathbb{N}$ so that $(U \wedge I)(\mathbf{1}) = 0$. If $0 \leq f \leq \alpha\mathbf{1}$ then $0 \leq (U \wedge I)(f) \leq \alpha(U \wedge I)(\mathbf{1}) = 0$ so that $(U \wedge I)(f) = 0$. $\square$

It would be possible to complete the result in Theorem 7.1 for unital representations on Dedekind complete Banach lattices if we had an answer to the following question.

Question 7.3. If E is a Dedekind complete Banach lattice then the *ideal centre* of E, $Z(E)$, is the lattice ideal generated in $\mathcal{L}^r(E)$ by the identity operator I_E. Actually $Z(E)$ is a projection band in $\mathcal{L}^r(E)$. If $\mathcal{P}$ denotes the band projection of $\mathcal{L}^r(E)$ onto $Z(E)$, T is a positive projection on E and $\mathcal{P}(T) = \alpha I_E$, what values can α take? We have shown that, for finite dimensional E, $\alpha \in \{n^{-1} : n \in \mathbb{N}\} \cup \{0\}$. Are any other values possible?

The left regular representation of $(\mathbb{R}^2, \times, P^1_\alpha)$ is easily checked to be a faithful unital lattice representation when $\alpha = -1$. Theorem 7.1 tells us that for $-1 < \alpha \leq 0$, as the left regular representation is unital, it has too small a dimension to be a unital lattice representation, even if one exists at all.

There remain only to consider the cases $A = (\mathbb{R}^2, \times, P^\beta_\alpha$ for $0 < \beta < 1$, when the multiplicative identity does not lie in the positive cone. We are led to believe, after conducting computerized searches, that there are no cases where there is a lattice representation in (at least) finite dimensional spaces. We are, unfortunately, unable to prove this. One approach that might be considered would be to add a positive identity and then try to find a unital lattice representation of A_e which can be restricted to a non-unital lattice representation of A. That approach is doomed to failure, as far as seeking finite dimensional representations is concerned, as we can show that there cannot be a finite dimensional unital lattice representation of A_e. In fact the following result probably precludes the existence of finite dimensional representations of many Riesz algebras.

Proposition 7.4. *Let A be unital Riesz algebra with positive identity e and a positive idempotent p that is disjoint from e, then there is no faithful finite dimensional unital lattice representation of A.*

Proof. If π is a faithful unital lattice representation of A in $\mathcal{L}^r(\mathbb{R}^n)$ then $\pi(e)$ is the identity $n \times n$ matrix and $P = \pi(p)$ is a non-zero positive idempotent matrix. In the final paragraph of the proof of Theorem 7.1 we pointed out that P has a strictly positive diagonal element so that P is not disjoint from the identity matrix, contradicting π being a lattice representation. $\square$

Corollary 7.5. *If A is any unital two-dimensional Riesz algebra then there is no faithful finite dimensional lattice representation of A_e.*

Proof. All of these algebras contain a positive idempotent and the added positive multiplicative identity will be disjoint from that. $\square$

Proposition 7.2 shows that there is no analogue of this corollary for infinite dimensional representations.

8. Norms

We commenced this study in an attempt to understand a very simple collection of Banach lattice algebras, thinking that two dimensional examples would be fairly simple to get to grips with. The diversity of possibilities that we have encountered shows how naive he was! The reader might think that some of the problems arise because our algebras cannot be normed in a sufficiently nice manner. That is not the case. By a *Banach lattice algebra* we will mean a Riesz algebra with a norm on it which is a lattice norm and which is sub-multiplicative. We will not assume that the identity has norm one. We will show that all our examples of two dimensional unital Riesz algebras may be normed as Banach lattice algebras. Theorem 2 of [3] tells us that if a Banach lattice algebra has a multiplicative identity with norm one then it must be positive. That means that our examples $(\mathbb{R}^2, \times_{\mathbb{D}}, P_1^\infty)$ and $(\mathbb{R}^2, \times, P_\alpha^\beta)$ for $\beta < 1$ certainly cannot be given a Banach lattice algebra norm with the identity having norm one, but in all other cases that is possible.

If we give $\mathbb{R}^n$ any lattice norm and then give $\mathcal{L}^r(\mathbb{R}^n)$ the corresponding regular norm then $\mathcal{L}^r(\mathbb{R}^n)$ becomes a Banach lattice algebra with the identity operator having norm one. The regular norm is defined by $\|T\|_r = \big\||T|\big\|$, where $|T|$ is the modulus of T in the Riesz space $\mathcal{L}^r(\mathbb{R}^n)$. The matrix that represents T is obtained by taking the matrix representing T and then taking the modulus of every entry. If there is a unital lattice representation of A in some $\mathcal{L}^r(\mathbb{R}^n)$ then taking any lattice norm on $\mathbb{R}^n$ we may identify A with a sublattice and unital subalgebra of $\mathcal{L}^r(\mathbb{R}^n)$ which will make it a Banach lattice algebra with the multiplicative identity having norm one. In particular this means that, in view of Proposition 6.1, $(\mathbb{R}^2, \times_{\mathbb{D}}, P_0^\infty)$ may be given (uncountably many equivalent) Banach lattice algebra norms with the multiplicative identity having norm one. Similar reasoning, using Proposition 6.3, shows that $(\mathbb{R}^2, \times_{\mathbb{D}}, P_1^\infty)$ may be given Banach lattice algebra norms. This time the identity is not represented by the identity operator and will not have norm one.

Turning now to the split complex numbers, or more specifically $\mathbb{R}^2$ with coordinate-wise multiplication, the simple case $(\mathbb{R}^2, \times, P_0^\infty)$ is clearly a Banach lattice algebra under the supremum norm with the identity having norm one. We turn now to the algebras $(\mathbb{R}^2, \times, P_\alpha^1)$ for $-1 \le \alpha \le 0$. Obtaining a simple description of the lattice structure helps us here, so we make a change of basis.

Proposition 8.1. *The two Riesz algebras* $A = (\mathbb{R}^2, \times, P^1_\alpha)$ *and* $B = (\mathbb{R}^2, \star_\alpha, P^\infty_0)$ *are isomorphic, where*

$$(a, b) \star_\alpha (a', b') = \big(aa' - \alpha bb', ab' + a'b + (1 - \alpha)bb'\big).$$

Proof. Let $u = (1, 1)$ and $v = (1, \alpha)$ in A and define $\pi : B \to A$ by $\pi(a, b) = au + bv$. As π is a linear bijection and maps two disjoint non-zero extremal elements of the posiitve cone in B to disjoint non-zero extremal elements of the positive cone in A, it is an order isomorphism. We must verify that π respects the multiplications.

$$\begin{aligned}\pi(a, b) \times \pi(a', b') &= (a + b, a + \alpha b)(a' + b', a' + \alpha b')\\ &= \big((a + b)(a' + b'), (a + \alpha b)(a' + \alpha b')\big)\\ &= (aa' - \alpha bb')u + \big(a'b + a'b + (1 + \alpha)bb')v\\ &= \pi\big((a, b) \star_\alpha (a', b')\big).\end{aligned}$$

Furthermore, the identity in B is easily checked to be $(1, 0)$ whilst $\pi(1, 0) = (1, 1)$ is the identity in A. ☐

Proposition 8.2. *There is a Banach lattice algebra norm on* B*, and therefore on* A*, for which the multiplicative identity has norm one.*

Proof. We claim that the norm $\|(a, b)\|_1 = |a| + |b|$ on B is the required norm. It is clearly a lattice norm and the identity has norm one, so we need only prove that it is sub-multiplicative. This is simple to verify as

$$\begin{aligned}\|(a, b) \star_\alpha (a', b')\|_1 &= |aa' - \alpha bb'| + |ab' + a'b + (1 - \alpha)bb'|\\ &\le |a||a'| + |\alpha||b||b'| + |a||b'| + |a'||b| + |1 - \alpha||b||b'|\\ &= (|a| + |b|)(|a'| + |b'|) = \|(a, b)\|_1\|(a', b')\|_1,\end{aligned}$$

when we recall that $-1 \le \alpha \le 0$ so that $|\alpha| + |1-\alpha| = (-\alpha) + (1+\alpha) = 1$. ☐

Finally, we have:

Proposition 8.3. *There is a Banach lattice algebra norm on* $(\mathbb{R}^2, \times, P^\beta_\alpha)$ *whenever* $0 < \beta < 1$ *and* $-\sqrt{\beta} \le \alpha \le 0$.

Proof. The proof is similar to that of the preceding proposition so we omit details. This time the isomorphic algebra is $(\mathbb{R}^2, \star^\beta_\alpha, P^\infty_0)$ where

$$\begin{aligned}(a, b)\star^\beta_\alpha(a', b') = (\beta - \alpha)^{-1}&\times\\ \big((\beta^2 - \alpha)aa' &+ \alpha(\beta - 1)(ab' + a'b) + (\alpha^2 - \alpha)bb',\\ &\beta(1 - \alpha)(ab' + a'b) + (\beta - \alpha^2)bb'\big).\end{aligned}$$

Putting the $\|\cdot\|_1$ norm on this algebra gives a lattice norm with $\|(a,b)\star_\alpha^\beta (a',b')\| \leq K\|(a,b)\|_1\|(a',b')\|_1$ where K is a certain strictly positive real constant depending on α and β. The norm $\|(a,b)\|' = K\|(a,b)\|_1$ remains a lattice norm and is now sub-multiplicative. Of course, as we must expect, the identity will not have norm one. □

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